

ASSOCIATED PRIMES OF POWERS OF EDGE IDEALS AND EAR DECOMPOSITIONS OF GRAPHS

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ABSTRACT. To compute the local cohomology of a monomial ideal one needs to know its associated primes. In this paper, we give an explicit description of the associated primes of every power of the edge ideal in terms of ear decompositions of the graph. This result establishes a surprising relationship between two seemingly unrelated notions of Commutative Algebra and Combinatorics. It covers all previous major results in this topic and has several interesting consequences in both fields.

INTRODUCTION

This work is motivated by the asymptotic properties of powers of a graded ideal Q in a graded algebra S over a field. It is known that for t sufficiently large, the depth of R/Q^t is a constant [3] and the Castelnuovo-Mumford regularity of Q^t is a linear function [6], [22]. These results have led to recent works on the behavior of the whole functions $\text{depth } S/Q^t$ [15] and $\text{reg } Q^t$ [8], [9]. Inevitably, one has to address the problem of estimating $\text{depth } S/Q^t$ and $\text{reg } Q^t$ for initial values of t . This problem is hard because there is no general approach to study a particular power Q^t . One may only hope to solve this problem for ideals which possess additional structures.

Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field k . Given a simple graph Γ on the vertex set $V = \{1, \dots, n\}$, one calls the ideal I generated by the monomials $x_i x_j$, $\{i, j\} \in \Gamma$, the *edge ideal* of Γ . This notion and its generalization to hypergraphs have provided many interesting correspondences between Combinatorics and Commutative Algebra (see e.g. [16], [14], [26], [30], [32]). Until now there were only a few works on $\text{depth } R/I^t$ and $\text{reg } I^t$ for special classes of edge ideals or on their asymptotic behavior [1], [19], [25], [27], [34].

It is known that the depth and the Castelnuovo-Mumford regularity can be defined in terms of the local cohomology modules. By a result of Takayama [31], the local cohomology modules of R/I^t can be computed by means of the reduced homology of certain simplicial complexes. The facets of these simplicial complexes correspond to associated primes of I^t [33]. Therefore, it is necessary to know the associated primes

2000 *Mathematics Subject Classification.* 13C05, 05C70.

Key words and phrases. edge ideal, graph, associated prime, index of stability, socle, weighted graph, matching number, dominating set, matching-critical, factor-critical, ear decomposition, non-bipartite graph.

This work is supported by the National Foundation for Science and Technology Development. Part of this work was carried out when the authors were visiting Vietnam Institute for Advanced Study in Mathematics in 2014.

of I^t in order to compute $\text{depth } R/I^t$ and $\text{reg } I^t$. In particular, $\text{depth } R/I^t > 0$ if and only if \mathfrak{m} is not an associated prime of I^t , where \mathfrak{m} denotes the maximal homogeneous ideal of R .

The associated prime of I^t has the form $P_F = (x_i \mid i \in F)$, where F is a (vertex) cover of Γ . In particular, the minimal associated primes of I^t correspond to the minimal covers of Γ . The problem is for which cover F is P_F a non-minimal associated prime of I^t . For simplicity we will call such a prime an *embedded prime* of I^t .

In a pioneering paper on edge ideals [30], Simis, Vasconcelos, and Villarreal showed that I^t has no embedded primes for all t if and only if Γ is a bipartite graph. If Γ is non-bipartite, Chen, Morey and Sung [5] gave an algorithm to detect the embedded primes of I^t for t sufficiently large. Associated primes of powers of squarefree monomial ideals were also studied by Ha and Morey [13] and Francisco, Ha and Van Tuyl [10]. However, these works could not be used to describe the embedded primes of every I^t in terms of Γ .

For $t = 2$, the above problem was solved by Terai and Trung [33]. They showed that P_F is an embedded prime of I^2 if and only if F is minimal among the covers containing the neighborhood of a triangle. A somewhat weaker result was obtained independently by Herzog and Hibi in [17], where it is proved that \mathfrak{m} is an associated prime of I^2 if and only if Γ has a dominating triangle. Recently, Hien, Lam and Trung [21] and Hien and Lam [20] succeeded in giving a complete classification of the embedded primes of I^3 and I^4 in terms of Γ . These classifications are so complicated that a classification for every $t \geq 4$ seems to be impossible.

In this paper we surprisingly obtain an explicit description of the embedded primes of I^t for every $t \geq 2$ by using ear decompositions of graphs. An *ear* is a path or a cycle with a specified endpoint. For convenience we allow 2-cycles with a repetition of an edge. An *ear decomposition* of a graph is a sequence of ears involving all vertices such that the first ear is a cycle and the endpoints of each subsequent ear are the only vertices belonging to earlier ears in the sequence. If the first ear is an odd cycle, we call it an *initially odd ear decomposition*. Non-bipartite connected graphs always have initially odd ear decompositions. We call a graph *strongly non-bipartite* if every connected component is non-bipartite. For a strongly non-bipartite graph Γ we introduce the invariant

$$\mu^*(\Gamma) := (\varphi^*(\Gamma) + n - c)/2,$$

where $\varphi^*(\Gamma)$ is the minimal number of even ears in families of initially odd ear decompositions of the connected components and c is the number of the connected components of Γ . With these notions, our main result can be stated as follows.

Theorem 4.3. Let F be a cover of Γ . Then P_F is an associated prime of I^t if and only if F is a minimal cover or F is minimal among the covers containing the closed neighborhood of a subset $U \subseteq V$ such that the induced subgraph Γ_U is strongly non-bipartite with $\mu^*(\Gamma_U) \leq t - 1$.

This result provides a simple way to determine all embedded primes of every power I^t . In fact, one only needs to look for strongly non-bipartite induced subgraphs of Γ in order to find such primes.

Example. Let Γ be the graph in Figure 1. Then Γ has only two odd cycles on the sets $U_1 = \{1, 2, 3\}$ and $U_2 = \{2, 3, 4, 5, 6\}$. Hence, any subset $U \subseteq V$ such that the induced subgraph Γ_U is strongly non-bipartite must contain U_1 or U_2 . The closed neighborhood of such a set U is either $F_1 = \{1, 2, 3, 4, 5\}$ or $F_2 = \{1, 2, 3, 4, 5, 6\} = V$, which are covers of Γ . Since $\mu^*(\Gamma_{U_1}) = 1$ and $\mu^*(\Gamma_{U_2}) = 2$, we can conclude that I^2 has one embedded prime (x_1, \dots, x_5) , and I^t , $t \geq 3$, have two embedded primes (x_1, \dots, x_5) and (x_1, \dots, x_6) .

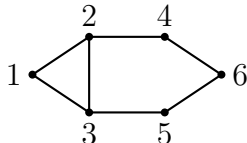


FIGURE 1.

The afore mentioned result of Vasconcelos, and Villarreal [30] immediately follows from Theorem 4.3. Since the triangle is the only strongly non-bipartite graph with $\mu^*(\Gamma) = 1$, the same can be said about the results in the case $t = 2$ of Herzog and Hibi [17], Terai and Trung [33]. Using a recursive description of strongly non-bipartite graphs with a given μ^* -invariant, we can easily classify the embedded primes of I^t for $t = 3$ and so on.

Let $\text{Ass}(I^t)$ denote the associated primes of I^t . From Theorem 4.3 we can immediately see that $\text{Ass}(I^t) \subseteq \text{Ass}(I^{t+1})$ for all $t \geq 1$. This property was a major result on $\text{Ass}(I^t)$ obtained recently by Martinez-Bernal, Morey and Villarreal [24]. It is known by Brodmann [2] that there is an integer t_0 such that $\text{Ass}(I^t) = \text{Ass}(I^{t+1})$ for all $t \geq t_0$. Let $\text{Ass}^\infty(I)$ denote the stable set $\text{Ass}(I^{t_0})$ and $\text{astab}(I)$ the index of stability of $\text{Ass}(I^t)$, which is the least number t_0 with this property. Theorem 4.3 shows that $\text{Ass}^\infty(I)$ is the set of the primes P_F , where F is a minimal cover or F is minimal among the covers containing the closed neighborhood of a subset $U \subseteq V$ with Γ_U strongly non-bipartite. This gives a complete description of Ass^∞ in comparison to the algorithm for computing $\text{Ass}^\infty(I)$ by Chen, Morey and Sung [5]. Furthermore, we can give formulas for the index of stability of every associated prime of $\text{Ass}^\infty(I)$ and for $\text{astab}(I)$, thereby answering a question raised by Sharp [29] for edge ideals.

The basic idea for our result can be found already in [21], where it is shown that a prime P_F is an embedded prime of I^t if and only if F is minimal among the covers of Γ containing the closed neighborhood of a set $U \subseteq V$ such that there exists a (vertex) weighted graph on U with some matching properties. However, these matching properties are so complicated that a characterization of the base graph Γ_U seems to be out of reach. Our new idea is that one can replace these weighted graphs by matching-critical weighted graphs. A graph Γ is called *matching-critical* if $\nu(\Gamma) = \nu(\Gamma - i)$ for all $i \in V$, where $\nu(\Gamma)$ denotes the matching number of Γ . It is known by results of Gallai [12] and Lovazs [23] that a connected graph is matching-critical if and only if it has an ear decomposition whose ears are odd. These results can be extended to weighted graphs with some modifications, and they will play a

crucial role in our investigation. It turns out that there exists a matching-critical weighted graph on U with matching number t if and only if Γ_U is strongly non-bipartite with $\mu^*(\Gamma_U) \leq t - 1$, leading to our main result. We would like to mention that matching-critical graphs have been studied independently by H. L. Truong [35] by means of irreducible decompositions of powers of the edge ideal.

Our description of the associated primes of I^t is only the first step in the estimation of depth R/I^t and $\text{reg } I^t$. The next step is to describe the simplicial complexes appearing in Takayama's formula for the local cohomology modules of R/I^t , which, we hope, could be solved in a satisfactory manner by using the results of this paper. Another problem is whether one can find a similar characterization of the associated primes of I^t when I is the edge ideal of a hypergraph.

Our investigation has also raised interesting problems of pure combinatorial nature. For instance, $\mu^*(\Gamma)$ is inspired of the invariant $\mu(\Gamma)$ in graph theory, which is the maximal cardinality of a join of a connected graph Γ . Frank [11] showed that

$$\mu(\Gamma) = (\varphi(\Gamma) + n - 1)/2,$$

where $\varphi(\Gamma)$ denotes the minimal number of even ears in ear decompositions of Γ . These invariants have several interesting meanings in combinatorics. We don't know whether $\mu^*(\Gamma) = \mu(\Gamma)$ for a non-bipartite graph Γ or whether $\mu^*(\Gamma)$ can be defined in a similar way as above.

The paper is organized as follows. In Section 1 we describe the monomials of the socle of the ring R/I^t in terms of weighted graphs. The existence of such a monomial is a criterion for $\mathfrak{m} \in \text{Ass}(I^t)$. This description leads to the study of matching-weighted graphs and their relationship with ear decompositions in Section 2. In Section 3 we use ear-decompositions to characterize the base graphs of matching-weighted graphs with a given matching number. The characterization of the embedded primes of I^t and its applications are given in Section 4. We conclude the paper with Section 5, where we recursively describe the minimal strongly non-bipartite graphs with a given μ^* -invariant and show how to use it to classify the associated primes of I^t .

Throughout this paper, Γ denotes a simple graph on the vertex set $V = \{1, \dots, n\}$ and I is the edge ideal of Γ in the polynomial ring $R = k[x_1, \dots, x_n]$. For unexplained terminology in commutative algebra and in graph theory we refer to [7] and [36], respectively.

The authors are thankful to Amin Seyed Fakhari, Dariush Kiani and Sara Saeedi Madani for pointing out that the induced subgraphs considered in this paper should have no isolated vertices.

1. SOCLE OF POWERS OF EDGE IDEALS

We will keep the notations of the introduction. First, we study the problem when $\mathfrak{m} \in \text{Ass}(I^t)$. It is well known that $\mathfrak{m} \in \text{Ass}(I^t)$ if and only if $(I^t : \mathfrak{m})/I^t \neq 0$. The ideal $(I^t : \mathfrak{m})/I^t$ is called the *socle* of the quotient ring R/I^t .

For a vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ we denote by $x^{\mathbf{a}}$ the monomial $x_1^{a_1} \cdots x_n^{a_n}$. Since I^t and $I^t : \mathfrak{m}$ are monomial ideals, $(I^t : \mathfrak{m})/I^t \neq 0$ if and only if there exists a

monomial $x^{\mathbf{a}} \in (I^t : \mathfrak{m}) \setminus I^t$. This condition means $x^{\mathbf{a}} \notin I^t$ and $x^{\mathbf{a}+\mathbf{e}_i} \in I^t$ for all $i \in V$, where \mathbf{e}_i denotes the i -th unit vector in \mathbb{N}^n .

It is easy to see that $x^{\mathbf{a}} \in I^t$ if and only if there is a family of t not necessarily different edges $\{i_1, j_1\}, \dots, \{i_t, j_t\}$ of Γ such that

$$\mathbf{a} \geq \sum_{r=1}^t (\mathbf{e}_{i_r} + \mathbf{e}_{j_r})$$

componentwise. This condition means the component a_i is greater than or equal to the appearing times of the vertex i in these edges for all $i = 1, \dots, n$.

Let $\Gamma_{\mathbf{a}}$ denote the weighted graph on Γ in which every vertex i is assigned the weight a_i , $i = 1, \dots, n$. Note that Γ is also a weighted graph with the weights $a_i = 1$. A *matching* of the weighted graph $\Gamma_{\mathbf{a}}$ is a family of not necessarily different edges of Γ such that a_i is greater than or equal to the number of appearing times of the vertex i in these edges. The largest possible number of edges in a matching of $\Gamma_{\mathbf{a}}$ is called the *matching number* of $\Gamma_{\mathbf{a}}$ and denoted by $\nu(\Gamma_{\mathbf{a}})$. It is clear that these notions coincide with the corresponding notions for a graph.

By the above observation we immediately obtain the following criterion for a monomial in I^t .

Lemma 1.1. $x^{\mathbf{a}} \in I^t$ if and only if $\nu(\Gamma_{\mathbf{a}}) \geq t$.

When dealing with matchings of the weighted graph $\Gamma_{\mathbf{a}}$ we can pass to a simple graph $p(\Gamma_{\mathbf{a}})$ on $a_1 \cdots a_n$ new vertices $1_1, \dots, 1_{a_1}, \dots, n_1, \dots, n_{a_n}$, where $\{i_r, j_s\}$ is an edge of $p(\Gamma_{\mathbf{a}})$ if and only if $\{i, j\}$ is an edge of Γ . In other words, we replace each vertex i of Γ by a_i new vertices i_1, \dots, i_{a_i} and each edge $\{i, j\}$ of Γ by $a_i a_j$ new edges $\{i_r, j_s\}$, $r = 1, \dots, a_i, j = 1, \dots, a_j$.

We call $p(\Gamma_{\mathbf{a}})$ the *polarization* of $\Gamma_{\mathbf{a}}$. The name originates from the fact that the edge ideal of $p(\Gamma_{\mathbf{a}})$ is a polarization of the ideal I (see e.g. [4], [18] for the definition of a polarization). The polarizations of a weighted graph are usually very large graphs. However, they are suited for visualizing weighted graphs when the weights are small.

Example 1.2. Let Γ be the following graph on 4 vertices and $\mathbf{a} = (1, 1, 2, 1)$. Then $\Gamma_{\mathbf{a}}$ and $p(\Gamma_{\mathbf{a}})$ can be illustrated as in Figure I.

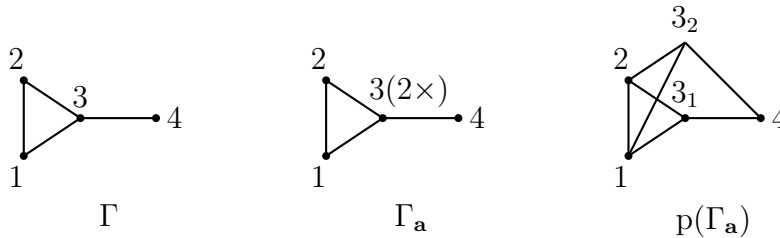


FIGURE 2.

We call the map which sending every edge $\{i_r, j_s\}$ of $p(\Gamma_{\mathbf{a}})$ to the edge $\{i, j\}$ of $\Gamma_{\mathbf{a}}$ the *projection* of $p(\Gamma_{\mathbf{a}})$ to $\Gamma_{\mathbf{a}}$. By this map, we obtain from a matching of $p(\Gamma_{\mathbf{a}})$

a matching of $\Gamma_{\mathbf{a}}$ of the same cardinality. Conversely, for every matching M of $\Gamma_{\mathbf{a}}$ we can find a matching M^* of $p(\Gamma_{\mathbf{a}})$ of the same cardinality such that M is the projection of M^* . This follows from the fact that for every $i \in V$, a_i is the number of vertices of $p(\Gamma_{\mathbf{a}})$ projected to i and that a_i is greater than or equal to the appearing times of the vertex i in M . As an immediate consequence, $\Gamma_{\mathbf{a}}$ and $p(\Gamma_{\mathbf{a}})$ have the same matching number.

Lemma 1.3. $\nu(\Gamma_{\mathbf{a}}) = \nu(p(\Gamma_{\mathbf{a}}))$.

Let $V(\mathbf{a}) := \{i \in V \mid a_i \neq 0\}$ be the support of \mathbf{a} . Then $V(\mathbf{a})$ is the vertex set of $\Gamma_{\mathbf{a}}$ since the edges of $\Gamma_{\mathbf{a}}$ do not involve any vertex i with weight $a_i = 0$. We will see that the condition $x^{\mathbf{a}} \in (I^t : \mathfrak{m}) \setminus I^t$ imposes strong conditions on $V(\mathbf{a})$ and $\Gamma_{\mathbf{a}}$, which can be used to give a criterion for $\mathfrak{m} \in \text{Ass}(I^t)$.

For a subset U of V we denote by $N[U]$ the *closed neighborhood* of U in Γ , i.e. the union of U and the set of the vertices adjacent to vertices of U . If $V = N[U]$, one calls U a *dominating set* of Γ .

Proposition 1.4. *Assume that $x^{\mathbf{a}} \in (I^t : \mathfrak{m}) \setminus I^t$, $t \geq 2$. Then*

- (i) $V(\mathbf{a})$ is a dominating set of Γ ,
- (ii) $\Gamma_{\mathbf{a}}$ has no isolated vertices,
- (iii) $\nu(\Gamma_{\mathbf{a}}) = t - 1$,
- (iv) *either $\nu(\Gamma_{\mathbf{a}-\mathbf{e}_i}) = t - 1$ for all $i \in V(\mathbf{a})$ or $x^{\mathbf{a}-\mathbf{e}_i} \in (I^{t-1} : \mathfrak{m}) \setminus I^{t-1}$ for some $i \in V(\mathbf{a})$.*

Proof. The assumption means $x^{\mathbf{a}} \notin I^t$ and $x^{\mathbf{a}+\mathbf{e}_i} \in I^t$ for all $i \in V$. By Lemma 1.1, these conditions can be expressed as $\nu(\Gamma_{\mathbf{a}}) < t$ and $\nu(\Gamma_{\mathbf{a}+\mathbf{e}_i}) \geq t$ for all $i \in V$.

(i) We have $\nu(\Gamma_{\mathbf{a}+\mathbf{e}_i}) \geq t > \nu(\Gamma_{\mathbf{a}})$ for all $i \in V$. From this it follows that if $i \notin V(\mathbf{a})$, $\Gamma_{\mathbf{a}+\mathbf{e}_i}$ must have at least one edge containing i . Hence i is adjacent to a vertex of $V(\mathbf{a})$. Thus, $V = N[V(\mathbf{a})]$.

(ii) If $\Gamma_{\mathbf{a}}$ has an isolated vertex i , then $\nu(\Gamma_{\mathbf{a}+\mathbf{e}_i}) = \nu(\Gamma_{\mathbf{a}})$, which contradicts the above conditions.

(iii) By Lemma 1.3, $\nu(\Gamma_{\mathbf{a}}) = \nu(p(\Gamma_{\mathbf{a}}))$ and $\nu(\Gamma_{\mathbf{a}+\mathbf{e}_i}) = \nu(p(\Gamma_{\mathbf{a}+\mathbf{e}_i}))$ for all $i \in V$. Since $p(\Gamma_{\mathbf{a}})$ can be obtained from $p(\Gamma_{\mathbf{a}+\mathbf{e}_j})$ by deleting a vertex, $\nu(p(\Gamma_{\mathbf{a}})) \geq \nu(p(\Gamma_{\mathbf{a}+\mathbf{e}_i})) - 1$. Therefore, $t - 1 \geq \nu(\Gamma_{\mathbf{a}}) \geq \nu(\Gamma_{\mathbf{a}+\mathbf{e}_i}) - 1 \geq t - 1$. This implies $\nu(\Gamma_{\mathbf{a}}) = t - 1$.

(iv) Since every matching of $\Gamma_{\mathbf{a}-\mathbf{e}_i}$ is also a matching of $\Gamma_{\mathbf{a}}$, we have $\nu(\Gamma_{\mathbf{a}-\mathbf{e}_i}) \leq \nu(\Gamma_{\mathbf{a}}) = t - 1$ for $i \in V(\mathbf{a})$. If $\nu(\Gamma_{\mathbf{a}-\mathbf{e}_i}) \geq t - 1$, this implies $\nu(\Gamma_{\mathbf{a}-\mathbf{e}_i}) = t - 1$. Similarly, we also have $\nu(\Gamma_{\mathbf{a}-\mathbf{e}_i+\mathbf{e}_j}) \geq \nu(\Gamma_{\mathbf{a}+\mathbf{e}_j}) - 1 \geq t - 1$ for all $j \in V$. By Lemma 1.1, this implies $x^{\mathbf{a}-\mathbf{e}_i+\mathbf{e}_j} \in I^{t-1}$. If $\nu(\Gamma_{\mathbf{a}-\mathbf{e}_i}) < t - 1$, we also have $x^{\mathbf{a}-\mathbf{e}_i} \notin I^{t-1}$ by Lemma 1.1. Therefore, $x^{\mathbf{a}-\mathbf{e}_i} \in (I^{t-1} : \mathfrak{m}) \setminus I^{t-1}$ in this case. \square

Proposition 1.5. *Assume that $\mathfrak{m} \in \text{Ass}(I^t) \setminus \text{Ass}(I^{t-1})$, $t \geq 2$. Then there exists a weighted graph $\Gamma_{\mathbf{a}}$ without isolated vertices such that $V(\mathbf{a})$ is a dominating set of Γ and $\nu(\Gamma_{\mathbf{a}-\mathbf{e}_i}) = \nu(\Gamma_{\mathbf{a}}) = t - 1$ for all $i \in V(\mathbf{a})$.*

Proof. As observed at the beginning of this section, the condition $\mathfrak{m} \in \text{Ass}(I^t)$ implies the existence of a monomial $x^{\mathbf{a}} \in (I^t : \mathfrak{m}) \setminus I^t$. Similarly, the condition

$\mathfrak{m} \notin \text{Ass}(I^{t-1})$ implies $I^{t-1} : \mathfrak{m} = I^{t-1}$. Therefore, there does not exist $i \in V(\mathbf{a})$ such that $x^{\mathbf{a}-\mathbf{e}_i} \in (I^{t-1} : \mathfrak{m}) \setminus I^{t-1}$. Hence, the conclusion follows from Proposition 1.4. \square

The following result is in a certain sense a converse to Proposition 1.5.

Proposition 1.6. *Assume that $V(\mathbf{a})$ is a dominating set of Γ , $\Gamma_{\mathbf{a}}$ has no isolated vertices, and $\nu(\Gamma_{\mathbf{a}-\mathbf{e}_i}) = \nu(\Gamma_{\mathbf{a}}) = t - 1$ for all $i \in V(\mathbf{a})$. Then $x^{\mathbf{a}} \in (I^t : \mathfrak{m}) \setminus I^t$.*

Proof. By Lemma 1.1, $\nu(\Gamma_{\mathbf{a}}) = t - 1$ implies $x^{\mathbf{a}} \notin I^t$. It remains to show that $x^{\mathbf{a}} \in I^t : \mathfrak{m}$ or, equivalently, $x^{\mathbf{a}+\mathbf{e}_i} \in I^t$ for all $i \in V$. Since $V(\mathbf{a})$ is a dominating set of $\Gamma_{\mathbf{a}}$ and $\Gamma_{\mathbf{a}}$ has no isolated vertices, we can find a vertex $j \in V(\mathbf{a})$ adjacent to i . Since $\nu(\Gamma_{\mathbf{a}-\mathbf{e}_j}) = t - 1$, $\Gamma_{\mathbf{a}-\mathbf{e}_j}$ has a matching of $t - 1$ edges. Adding the edge $\{i, j\}$ to this matching we obtain a matching of $\Gamma_{\mathbf{a}+\mathbf{e}_i}$. Thus, $\nu(\Gamma_{\mathbf{a}+\mathbf{e}_i}) \geq t$. By Lemma 1.1, this implies $x^{\mathbf{a}+\mathbf{e}_i} \in I^t$. \square

Remark 1.7. Let \tilde{I}^t denote the *saturation* of I^t . Then $\tilde{I}^t = \cup_{s \geq 0} I^t : \mathfrak{m}^s \supseteq I^t : \mathfrak{m}$. Since \tilde{I}^t/I^t is the local cohomology module $H_{\mathfrak{m}}^0(R/I^t)$, the condition $x^{\mathbf{a}} \in \tilde{I}^t \setminus I^t$ implies $a_i \leq t - 1$ for all $i \in V$ by a result of Takayama [31, Theorem 1] (see also [17, Proposition 1.1]). If $t = 2$, we have $a_i \leq 1$ for all $i \in V$. Hence $\Gamma_{\mathbf{a}}$ is the induced graph of Γ on $V(\mathbf{a})$. This explains why the case $t = 2$ can be handled in a relatively easy way [17, Theorem 2.1], [32, Theorem 2.8]. If $t \geq 3$, we need to investigate the vertex weighted graph $\Gamma_{\mathbf{a}}$, following the approach of [21]. The new idea here is that we take into account $\Gamma_{\mathbf{a}-\mathbf{e}_i}$. This idea allows us to relate the condition $x^{\mathbf{a}} \in (I^t : \mathfrak{m}) \setminus I^t$ to interesting notions in graph theory as we shall see in the next section.

2. MATCHING-CRITICAL WEIGHTED GRAPHS

In graph theory, one calls Γ a matching-critical graph if $\nu(\Gamma) = \nu(\Gamma - i)$ for every $i \in V$, where $\Gamma - i$ denotes the subgraph of Γ obtained by deleting the vertex i . Inspired by this notion, we call a weighted graph $\Gamma_{\mathbf{a}}$ *matching-critical* if $\nu(\Gamma_{\mathbf{a}-\mathbf{e}_i}) = \nu(\Gamma_{\mathbf{a}})$ for all $i \in V(\mathbf{a})$.

The matching-critical property is preserved by polarization.

Lemma 2.1. *$\Gamma_{\mathbf{a}}$ is a matching-critical weighted graph if and only if $p(\Gamma_{\mathbf{a}})$ is a matching-critical graph.*

Proof. By Lemma 1.3, $\nu(\Gamma_{\mathbf{a}}) = \nu(p(\Gamma_{\mathbf{a}}))$ and $\nu(\Gamma_{\mathbf{a}-\mathbf{e}_i}) = \nu(p(\Gamma_{\mathbf{a}-\mathbf{e}_i}))$. By the definition of a polarization, the graphs $p(\Gamma_{\mathbf{a}-\mathbf{e}_i})$ and $p(\Gamma_{\mathbf{a}}) - i_r$ are isomorphic for all $i \in V$ and $r = 1, \dots, a_i$. Hence $\nu(\Gamma_{\mathbf{a}}) = \nu(\Gamma_{\mathbf{a}-\mathbf{e}_i})$ for all $i \in V(\mathbf{a})$ if and only if $\nu(p(\Gamma_{\mathbf{a}})) = \nu(p(\Gamma_{\mathbf{a}}) - i_r)$ for all $i \in V(\mathbf{a})$ and $r = 1, \dots, a_i$. The latter condition just means that $p(\Gamma_{\mathbf{a}})$ is a matching-critical graph. \square

We say that $\Gamma_{\mathbf{a}}$ is a *connected* weighted graph if the induced subgraph $\Gamma_{V(\mathbf{a})}$ is connected. Let $\Gamma_1, \dots, \Gamma_c$ be the connected components of $\Gamma_{V(\mathbf{a})}$. Then there exists unique vectors $\mathbf{a}_1, \dots, \mathbf{a}_c \in \mathbb{N}^n$ such that $\mathbf{a} = \mathbf{a}_1 + \dots + \mathbf{a}_c$ and $\Gamma_i = \Gamma_{V(\mathbf{a}_i)}$, $i = 1, \dots, c$. We call the weighted graphs $\Gamma_{\mathbf{a}_1}, \dots, \Gamma_{\mathbf{a}_c}$ the *connected components* of $\Gamma_{\mathbf{a}}$.

Lemma 2.2. *A weighted graph is matching-critical if and only if its connected components are matching-critical.*

Proof. Let $\Gamma_{\mathbf{a}_1}, \dots, \Gamma_{\mathbf{a}_c}$ be the connected components of $\Gamma_{\mathbf{a}}$. It is easy to see that

$$\nu(\Gamma_{\mathbf{a}}) = \nu(\Gamma_{\mathbf{a}_1}) + \dots + \nu(\Gamma_{\mathbf{a}_c}).$$

For every $i \in V(\mathbf{a})$, there is a unique \mathbf{a}_r such that $i \in V(\mathbf{a}_r)$. Hence

$$\nu(\Gamma_{\mathbf{a}-\mathbf{e}_i}) = \nu(\Gamma_{\mathbf{a}_1}) + \dots + \nu(\Gamma_{\mathbf{a}_r-\mathbf{e}_i}) + \dots + \nu(\Gamma_{\mathbf{a}_c}).$$

Consequently, $\nu(\Gamma_{\mathbf{a}}) = \nu(\Gamma_{\mathbf{a}-\mathbf{e}_i})$ for all $i \in V(\mathbf{a})$ if and only if $\nu(\Gamma_{\mathbf{a}_r}) = \nu(\Gamma_{\mathbf{a}_r-\mathbf{e}_i})$ for all $i \in V(\mathbf{a}_r)$ and $r = 1, \dots, c$. \square

Connected matching-critical graphs have some remarkable characterizations. For instance, Gallai [12] showed that Γ is a matching-critical connected graph if and only if $\Gamma - i$ has a perfect matching for all $i \in V$. A graph with this property is called factor-critical [23].

We call a matching of a weighted graph $\Gamma_{\mathbf{a}}$ *perfect* if a_i is the appearing times of i in the edges of the matching for all $i \in V$. The weighted graph $\Gamma_{\mathbf{a}}$ is called *factor-critical* if $\Gamma_{\mathbf{a}-\mathbf{e}_i}$ has a perfect matching for all $i \in V(\mathbf{a})$. Obviously, this notion is a generalization of factor-critical graphs.

Proposition 2.3. *A connected weighted graph is matching-critical if and only if it is factor-critical.*

Proof. As we have seen in the previous section, there is a correspondence between matchings of a weighted graph and its polarization. By this correspondence, a weighted graph $\Gamma_{\mathbf{a}}$ is factor-critical if and only if $p(\Gamma_{\mathbf{a}-\mathbf{e}_i})$ has a perfect matching for all $i \in V(\mathbf{a})$. Since $p(\Gamma_{\mathbf{a}-\mathbf{e}_i})$ and $p(\Gamma_{\mathbf{a}}) - i_r$ are isomorphic for all $i \in V$ and $r = 1, \dots, a_i$, this is equivalent to the condition that $p(\Gamma_{\mathbf{a}})$ is factor-factorial. By the above mentioned result of Gallai, $p(\Gamma_{\mathbf{a}})$ is factor-factorial if and only if $p(\Gamma_{\mathbf{a}})$ is matching-critical. Hence the assertion follows from Lemma 2.1. \square

Corollary 2.4. *Let $\Gamma_{\mathbf{a}}$ be a matching-critical weighted graph and $|\mathbf{a}| = a_1 + \dots + a_n$. Then $\nu(\Gamma_{\mathbf{a}}) = (|\mathbf{a}| - c)/2$, where c is the number of connected components of Γ .*

Proof. By Lemma 2.2, every connected component of $\Gamma_{\mathbf{a}}$ is matching-critical. Since the matching number is additive on the connected components, we only need to show the assertion for the case $\Gamma_{\mathbf{a}}$ being connected. In this case, $\Gamma_{\mathbf{a}}$ is factor-critical by Proposition 2.3. By definition, $\Gamma_{\mathbf{a}-\mathbf{e}_1}$ has a perfect matching. Hence $\nu(\Gamma_{\mathbf{a}-\mathbf{e}_1}) = |\mathbf{a} - \mathbf{e}_1|/2$. Since $\nu(\Gamma_{\mathbf{a}-\mathbf{e}_1}) = \nu(\Gamma_{\mathbf{a}})$ and $|\mathbf{a} - \mathbf{e}_1| = |\mathbf{a}| - 1$, we obtain $\nu(\Gamma_{\mathbf{a}}) = (|\mathbf{a}| - 1)/2$. \square

By a beautiful result of Lovasz [23], a graph with more than one vertex is factor-critical if and only if it has an odd ear decomposition. Recall that an ear is a path or a cycle with a specified endpoint. An ear decomposition of a graph is a partition of the edges into a sequence of ears such that the first ear is a cycle and the endpoints of each subsequent ear are the only vertices belonging to earlier ears in the sequence. An ear decomposition is odd if all ears have odd lengths. To prove a similar result for weighted graphs, we need to modify these notions.

In the following we define an ear as a walk in the graph Γ . An ear decomposition of $\Gamma_{\mathbf{a}}$ is a sequence of ears such that the first ear is a closed walk, the endpoints of each subsequent ear belong to earlier ears, and, for all $i \in V$, a_i is the appearing times of i not as an endpoint of the subsequent ears in the decomposition. By this definition, an ear can be a 2-cycle with a repetition of an edge, i.e. a closed walk of length 2, which is not allowed in the old definition. Moreover, an ear decomposition of the new definition doesn't need to involve all edges.

For a simple graph, an ear decomposition of the old definition is an ear decomposition of the new definition. Conversely, if a simple graph has an ear decomposition of the new definition without 2-cycles, then every ear is an ear of the old definition since there are no repetitions of the inner vertices. Moreover, the vertices of every missing edge must appear already in the ears of the decomposition. Therefore, if we add the missing edges as ears of length one to the existing sequence of ears, we will get an ear decomposition of the old definition. As a consequence, a simple graph has an odd ear decomposition of the old definition if and only if it has an odd ear decomposition of the new definition. From this it follows that the above result of Lovasz remains valid for the new definition.

Example 2.5. Consider the graphs of Example 1.2. Then Γ has only an ear decomposition which starts with an odd cycle. This decomposition consists of the triangle $\{1, 2, 3\}$ and the 2-cycle of the edge $\{3, 4\}$. Hence Γ is not factor-critical. On the other hand, $\Gamma_{\mathbf{a}}$ is factor-critical because it has an odd ear decomposition consisting of only the closed walk $1, 2, 3, 4, 3, 1$. This can be also seen from the polarization $p(\Gamma_{\mathbf{a}})$, which has an odd ear decomposition consisting of only the cycle $1, 2, 3_1, 4, 3_2, 1$. If we consider the edges $\{1, 3_1\}$ and $\{2, 3_2\}$ as ears of length one and add them to this cycle, we obtain an odd ear decomposition of the old definition for $p(\Gamma_{\mathbf{a}})$.

Proposition 2.6. *A connected weighted graph with more than one vertex is factor-critical if and only if it has an odd ear decomposition.*

Proof. Let $\Gamma_{\mathbf{a}}$ be a connected weighted graph with more than one vertex. By Lemma 2.1 and Proposition 2.3, $\Gamma_{\mathbf{a}}$ is factor-critical if and only if $p(\Gamma_{\mathbf{a}})$ is factor-critical. By the afore mentioned result of Lovasz, $p(\Gamma_{\mathbf{a}})$ is factor-critical if and only if it has an odd ear decomposition. Therefore, it suffices to show that $p(\Gamma_{\mathbf{a}})$ has an odd ear decomposition if and only if $\Gamma_{\mathbf{a}}$ has an odd ear decomposition.

Assume that $p(\Gamma_{\mathbf{a}})$ has an odd ear decomposition \mathcal{E} . Projecting every edge $\{i_r, j_s\}$ of \mathcal{E} to the edge $\{i, j\}$ of Γ we obtain from every path of \mathcal{E} a walk in Γ of the same length. Obviously, these walks form an odd ear decomposition of $\Gamma_{\mathbf{a}}$.

Conversely, assume that $\Gamma_{\mathbf{a}}$ has an odd ear decomposition \mathcal{F} . We can associate with every ear of \mathcal{F} an ear of $p(\Gamma_{\mathbf{a}})$ such that its projection to Γ is the given ear of \mathcal{F} . Let \mathcal{E} be the sequence of these associated ears. The endpoints of the subsequent ears of \mathcal{E} can be clearly chosen to be vertices of earlier ears in \mathcal{E} . Since for every $i \in V$, there are a_i vertices of $p(\Gamma_{\mathbf{a}})$ which can be projected to i and a_i is the appearing times of i not as an endpoint of the subsequent ears in \mathcal{F} , we can choose the ears of \mathcal{E} such that every vertex of $p(\Gamma_{\mathbf{a}})$ appears once as a vertex of the first ear or as an

inner vertex of the subsequent ears of \mathcal{E} . Therefore, \mathcal{E} is an odd ear decomposition of $p(\Gamma_{\mathbf{a}})$. \square

Given a matching-critical weighted graph, we can easily construct a matching-critical weighted graph of any higher matching number on the same vertex set.

Corollary 2.7. *Let $\Gamma_{\mathbf{a}}$ be a matching-critical weighted graph. Let $\{i, j\}$ be an arbitrary edge of $\Gamma_{V(\mathbf{a})}$. Then $\Gamma_{\mathbf{a}+\mathbf{e}_i+\mathbf{e}_j}$ is a matching-critical weighted graph with $\nu(\Gamma_{\mathbf{a}+\mathbf{e}_i+\mathbf{e}_j}) = \nu(\Gamma_{\mathbf{a}}) + 1$.*

Proof. By Lemma 2.2, we may assume that $\Gamma_{\mathbf{a}}$ is connected. Then $\Gamma_{\mathbf{a}}$ has an odd ear decomposition \mathcal{E} by Proposition 2.6. Let C be an odd ear of \mathcal{E} containing the edge $\{i, j\}$. Let D be the odd ear obtained from C by replacing $\{i, j\}$ by the walk $\{i, j\}, \{j, i\}, \{i, j\}$. Replacing C by D we obtain from \mathcal{E} an odd ear decomposition of $\Gamma_{\mathbf{a}+\mathbf{e}_i+\mathbf{e}_j}$. Hence $\Gamma_{\mathbf{a}+\mathbf{e}_i+\mathbf{e}_j}$ is factor-critical by Proposition 2.6. By Corollary 2.4, we have

$$\nu(\Gamma_{\mathbf{a}+\mathbf{e}_i+\mathbf{e}_j}) = (|\mathbf{a} + \mathbf{e}_i + \mathbf{e}_j| - 1)/2 = (|\mathbf{a}| + 1)/2 = \nu(\Gamma_{\mathbf{a}}) + 1.$$

\square

Now we can give a criterion for $\mathbf{m} \in \text{Ass}(I^t)$ by means of matching-critical weighted graphs.

Theorem 2.8. *$\mathbf{m} \in \text{Ass}(I^t)$ if and only if there exists a matching critical weighted graph $\Gamma_{\mathbf{a}}$ without isolated vertices such that $V(\mathbf{a})$ is a dominating set of Γ and $\nu(\Gamma_{\mathbf{a}}) < t$.*

Proof. Since I is a radical ideal with $\dim R/I \geq 1$, $\mathbf{m} \notin \text{Ass}(I)$. Assume that $\mathbf{m} \in \text{Ass}(I^t)$, $t \geq 2$. Let s be the largest integer $< t$ such that $\mathbf{m} \notin \text{Ass}(I^s)$. Then $\mathbf{m} \in \text{Ass}(I^{s+1}) \setminus \text{Ass}(I^s)$. By Proposition 1.5, this condition implies the existence of a matching critical weighted graph $\Gamma_{\mathbf{a}}$ without isolated vertices such that $V(\mathbf{a})$ is a dominating set of Γ and $\nu(\Gamma_{\mathbf{a}}) = s$.

Conversely, assume that there exists a matching critical weighted graph $\Gamma_{\mathbf{a}}$ without isolated vertices such that $V(\mathbf{a})$ is a dominating set of Γ and $\nu(\Gamma_{\mathbf{a}}) < t$. Using Corollary 2.7, we can construct a matching critical weighted graph $\Gamma_{\mathbf{b}}$ without isolated vertices such that $V(\mathbf{b}) = V(\mathbf{a})$ and $\nu(\Gamma_{\mathbf{b}}) = t - 1$. By Proposition 1.6, this implies $\mathbf{m} \in \text{Ass}(I^t)$. \square

Remark 2.9. It is of great interest to find a similar criterion for $\mathbf{m} \in \text{Ass}(I^t)$ when I is the edge ideal of a hypergraph. In this general case, Proposition 1.5 still holds, whereas Proposition 1.6 does not. Anyway, one can define matching-critical weighted hypergraphs in the same way. As far as we know, matching-critical hypergraphs have not been studied in combinatorics before.

3. BASE GRAPHS

Our next goal is to deduce from Theorem 2.8 a criterion for $\mathbf{m} \in \text{Ass}(I^t)$ in terms of Γ without involving the existence of a matching weighted graphs $\Gamma_{\mathbf{a}}$. For that

purpose we have to investigate the induced subgraph $\Gamma_{V(\mathbf{a})}$ of Γ on $V(\mathbf{a})$, which we call the *base graph* of $\Gamma_{\mathbf{a}}$.

By Proposition 2.3 and Proposition 2.6, a connected weighted graph $\Gamma_{\mathbf{a}}$ with more than one vertex is matching-critical if and only if it has an odd ear decomposition. Since the graph $\Gamma_{V(\mathbf{a})}$ can be obtained from $\Gamma_{\mathbf{a}}$ by lowering the weights one by one, we want first to see what happen with an ear decomposition in passing from $\Gamma_{\mathbf{a}}$ to $\Gamma_{\mathbf{a}-\mathbf{e}_i}$ for $i \in V(\mathbf{a})$ with $a_i \geq 2$.

For an ear decomposition \mathcal{E} of a weighted graph we denote by $\varphi(\mathcal{E})$ the number of even ears in \mathcal{E} .

Lemma 3.1. *Let $\Gamma_{\mathbf{a}}$ be a connected weighted graph and \mathcal{E} an ear decomposition of $\Gamma_{\mathbf{a}}$. Let $i \in V(\mathbf{a})$ with $a_i \geq 2$. By breaking suitable walks of \mathcal{E} we can turn \mathcal{E} into an ear decomposition \mathcal{F} of $\Gamma_{\mathbf{a}-\mathbf{e}_i}$ such that*

- (i) $\varphi(\mathcal{F}) \leq \varphi(\mathcal{E}) + 1$,
- (ii) *If the first ear of \mathcal{E} is an odd cycle, then the first ear of \mathcal{F} is an odd cycle.*

Proof. Let C be the earliest ear of \mathcal{E} containing i . We distinguish the following cases.

Case 1: C passes i once. Since $a_i \geq 2$, there exists another ear of \mathcal{E} containing i as an inner vertex. Let C' be the next ear of \mathcal{E} containing i . Let u, v be the two endpoints of C' . Then i is different than u, v . Breaking C' at i into a walk D from u to i and a walk D' from i to v , we obtain from \mathcal{E} an ear decomposition \mathcal{F} of $\Gamma_{\mathbf{a}-\mathbf{e}_i}$ (see Figure 3). The number of the edges of C' is the sum of the number of the edges of D and D' . If C' is an odd ear, then either D or D' is an odd ear and the other is an even ear of \mathcal{F} . If C' is an even ear, then both D and D' are odd or even ears of \mathcal{E} . Therefore, we always have $\varphi(\mathcal{F}) \leq \varphi(\mathcal{E}) + 1$.

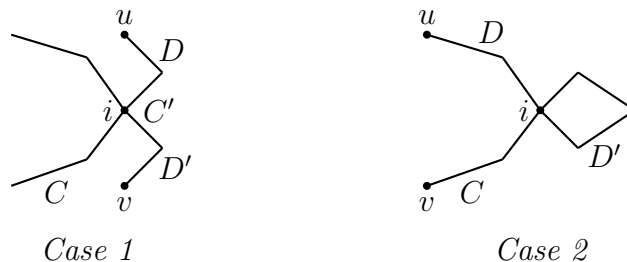


FIGURE 3.

Case 2: C passes i at least twice. Let u, v be the endpoints of C (which may be the same). If C is the first ear, then C is a closed walk. In this case, we may choose $u = v \neq i$. If C is not the first ear, i is different than u, v because u, v belong to earlier ears of \mathcal{E} . Breaking C at i into a walk D with endpoints u, v which passes i once and a closed walk D' ending at i , we obtain from \mathcal{E} an ear decomposition \mathcal{F} of $\Gamma_{\mathbf{a}-\mathbf{e}_i}$ (see Figure 3). Similarly as above, we can show that $\varphi(\mathcal{F}) \leq \varphi(\mathcal{E}) + 1$.

By the above arguments, the first ear of \mathcal{E} and \mathcal{F} are the same if C passes i once or if C is not the first ear. If C is the first ear passing i at least twice, then D and D' are closed walks. If C is an odd ear, one of the walks D and D' must be odd.

We choose this walk to be the first ear of \mathcal{F} . Thus, the first ear of \mathcal{F} is odd if the first ear of \mathcal{E} is odd. \square

We can use ear decompositions to give a characterization of the base graph of matching-critical weighted graphs with a given matching number. For that we shall need the following notions.

For an arbitrary graph Γ (which may be disconnected) we call a family of ear decompositions of the connected components an *ear decomposition* of Γ . If the ear decompositions of such a family begin with odd cycles, we call it an *initially odd ear decomposition*. We call Γ a *strongly non-bipartite* graph if every connected component is non-bipartite or, equivalently, if every connected component contains an odd cycle. It is easy to see that such a graph has at least one initially odd ear decomposition. We denote by $\varphi^*(\Gamma)$ the minimal total number of even ears in initially odd ear decompositions and define

$$\mu^*(\Gamma) = \frac{\varphi^*(\Gamma) + n - c}{2},$$

where c is the number of connected components of Γ . For instance, if Γ is an odd cycle of length $2t + 1$, then $\varphi^*(\Gamma) = 0$ and $\mu^*(\Gamma) = t$.

Theorem 3.2. Γ is the base graph of a matching-critical weighted graph $\Gamma_{\mathbf{a}}$ without isolated vertices with $\nu(\Gamma_{\mathbf{a}}) = t$ if and only if Γ is strongly non-bipartite with $\mu^*(\Gamma) \leq t$.

Proof. It is easy to see that the invariant $\mu^*(\Gamma)$ is additive on the connected components of Γ . Therefore, using Lemma 2.2 we only need to prove the assertion for a connected graph Γ with more than one vertex.

Assume that Γ is the base graph of a matching-critical weighted graph $\Gamma_{\mathbf{a}}$ with $\nu(\Gamma_{\mathbf{a}}) = t$. By Proposition 2.3, $\Gamma_{\mathbf{a}}$ is factor-critical. Hence $\Gamma_{\mathbf{a}}$ has an odd ear decomposition \mathcal{E} by Proposition 2.6. We have $\varphi(\mathcal{E}) = 0$. Since Γ is the base graph of $\Gamma_{\mathbf{a}}$, $V(\Gamma_{\mathbf{a}}) = V$. Hence $a_i \geq 1$ for all $i = 1, \dots, n$. Since $\Gamma = \Gamma_{(1, \dots, 1)}$ and since $(1, \dots, 1)$ can be obtained from \mathbf{a} by $|\mathbf{a}| - n$ subtractions by unit vectors, applying Lemma 3.1 successively we can see that Γ has an initially odd ear decomposition \mathcal{F} with $\varphi^*(\mathcal{F}) \leq |\mathbf{a}| - n$. This implies $\varphi^*(\Gamma) \leq |\mathbf{a}| - n$. Therefore, Γ is a non-bipartite graph and $\mu^*(\Gamma) \leq (|\mathbf{a}| - 1)/2 = \nu(\Gamma_{\mathbf{a}}) = t$ by Corollary 2.4.

Conversely, assume that Γ is a non-bipartite graph with $\mu^*(\Gamma) \leq t$. By Corollary 2.7, we only need to show that Γ is the base graph of a matching-critical weighted graph $\Gamma_{\mathbf{a}}$ with $\nu(\Gamma_{\mathbf{a}}) = \mu^*(\Gamma)$.

If $\varphi^*(\Gamma) = 0$, then Γ is a factor-critical graph by the afore mentioned result of Lovasz. Obviously, $\nu(\Gamma) = (n - 1)/2 = \mu^*(\Gamma)$.

If $\varphi^*(\Gamma) > 0$, we choose an initially odd ear decomposition \mathcal{E} of Γ with $\varphi(\mathcal{E}) = \varphi^*(\Gamma)$. For every even ear C of \mathcal{E} we fix an endpoint i . Let j be the first vertex of C adjacent to i and h an adjacent vertex of i in a preceding ear of C . Replacing $\{i, j\}$ by the walk $\{h, i\}, \{i, j\}$ we obtain from C an odd ear C' , where the weight of i increases by one. Let U be the set of the fixed endpoints. For every $i \in U$ let c_i be the number of even ears of \mathcal{E} whose fixed endpoint is i . Define a vector $\mathbf{a} \in \mathbb{N}^n$

by setting $a_i = c_i + 1$ if $i \in U$ and $a_i = 1$ if $i \notin U$. Replacing every even ears C of \mathcal{E} by an odd ears C' as above we obtain from \mathcal{E} an odd ear decomposition of the weighted graph $\Gamma_{\mathbf{a}}$. By Proposition 2.6, $\Gamma_{\mathbf{a}}$ is a factor-critical weighted graph. We have $|\mathbf{a}| = \sum_{i \in U} c_i + n = \varphi^*(\Gamma) + n$. Hence $\mu^*(\Gamma) = (|\mathbf{a}| - 1)/2$. By Corollary 2.4, this implies $\nu(\Gamma_{\mathbf{a}}) = \mu^*(\Gamma)$, as required. \square

Now we are able to give a criterion for $\mathbf{m} \in \text{Ass}(I^t)$ in terms of Γ without involving weighted graph.

Theorem 3.3. *$\mathbf{m} \in \text{Ass}(I^t)$ if and only if there exists a dominating set U of Γ such that Γ_U is strongly non-bipartite with $\mu^*(\Gamma_U) < t$.*

Proof. By Theorem 2.8, $\mathbf{m} \in \text{Ass}(I^t)$ if and only if there exists a matching-critical weighted graph $\Gamma_{\mathbf{a}}$ without isolated vertices such that $V(\mathbf{a})$ is a dominating set of Γ and $\nu(\Gamma_{\mathbf{a}}) < t$. By Theorem 3.2, this condition is satisfied if and only if $\Gamma_{V(\mathbf{a})}$ is strongly non-bipartite with $\mu^*(\Gamma_{V(\mathbf{a})}) < t$. \square

From Theorem 3.3 we immediately obtain the following results.

Corollary 3.4. *$\mathbf{m} \in \text{Ass}(I^t)$ for $t \gg 0$ if and only if Γ is strongly non-bipartite.*

This result can be also deduced from [30, Theorem 5.9] and [5, Lemma 2.1 and Corollary 3.4].

If Γ is strongly non-bipartite, we can compute the least number s such that $\mathbf{m} \in \text{Ass}(I^t)$ for $t \geq s$. For that we denote by $s(\Gamma)$ the minimum of $\mu^*(\Gamma_U)$, where U is a dominating set of Γ such that Γ_U is strongly non-bipartite.

Corollary 3.5. *Let Γ be a strongly non-bipartite graph. Then $\mathbf{m} \in \text{Ass}(I^t)$ if and only if $t \geq s(\Gamma) + 1$.*

Due to the above result it would be of interest to develop an effective method to compute $s(\Gamma)$. The following fact may be of some help in estimating $s(\Gamma)$.

Lemma 3.6. *Let Γ be a strongly non-bipartite graph. Let W denote the set of the vertices of degree ≥ 2 . Then W is a dominating set of Γ and Γ_W is a strongly non-bipartite graph with $\mu^*(\Gamma) = \mu^*(\Gamma_W) + n - |W|$.*

Proof. Since every connected component of Γ is not an edge, each vertex of degree one is not adjacent to a vertex of degree one. This means each vertex of $V \setminus W$ is adjacent to a vertex of W . Hence W is a dominating set of Γ . It is also clear that every connected component of Γ_W is the reduced graph of a connected component of Γ on W , which keeps the odd cycles of Γ . Therefore, Γ_W is a strongly non-bipartite graph.

To show that $\mu^*(\Gamma) = \mu^*(\Gamma_W) + n - |W|$, we only need to show that $\mu^*(\Gamma) = \mu^*(\Gamma - i) + 1$ for any vertex i of degree one. Let j be the vertex adjacent to i . It is clear that every ear decomposition of Γ must contain the 2-cycle of the edge $\{i, j\}$. From this it follows that $\varphi^*(\Gamma) = \varphi^*(\Gamma - i) + 1$, which implies the assertion. \square

Example 3.7. Let Γ be the graph in Figure 4. Then $W = \{1, 2, 3, 4, 5\}$ and any other dominating set U must contain W if Γ_U is strongly non-bipartite. By Lemma

3.6, $\mu^*(\Gamma_U) > \mu^*(\Gamma_W)$. Hence $s(\Gamma) = \mu^*(\Gamma_W) = 2$. Therefore, $\mathfrak{m} \in \text{Ass}(I^t)$ if and only if $t \geq 3$. On the other hand, it is easy to check that $s(\Gamma_W) = 1$. Since W is also the set of the vertices of degree ≥ 2 of Γ_W , this shows that in general, we may have $s(\Gamma) \neq \mu^*(\Gamma_W)$.

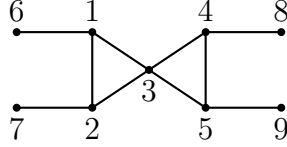


FIGURE 4.

4. ASSOCIATED PRIMES OF POWERS OF EDGE IDEALS

In this section we will characterize the associated primes of I^t in terms of Γ . For a subset F of V we set $P_F = (x_i \mid i \in F)$. It is well known that every associated prime of I^t is of the form P_F , where F is a (vertex) cover of Γ , i.e. F meets every edge of Γ . In particular, P_F is a minimal associated prime of I^t if and only if F is a minimal cover of Γ (see e.g. [18]). It remains to see when P_F is an embedded prime of I^t .

This problem can be reduced to the case $P_F = \mathfrak{m}$ by using the following notion. We define $c(F) = F \setminus N[V \setminus F]$ and call it the *core* of F [21]. In other words, $c(F)$ is the set of vertices in F which are not adjacent to any vertex in $V \setminus F$.

If F is a non-minimal cover of Γ , we can find a vertex $i \in F$ such that $F \setminus i$ is a cover of Γ . Therefore, for every edge $\{i, j\}$ we must have $j \in F$. This implies $i \in c(F)$. Hence $c(F) \neq \emptyset$.

Lemma 4.1. (cf. [21, Proposition 1.3]) *Let F be a non-minimal cover of Γ . Let $S = k[x_i \mid i \in c(F)]$. Let \mathfrak{n} be the maximal homogeneous ideal of S and J the edge ideal of the induced subgraph $\Gamma_{c(F)}$ in S . Then $P_F \in \text{Ass}(I^t)$ if and only if $\mathfrak{n} \in \text{Ass}(J^s)$ for some $s \leq t$.*

Proof. Let $A = R[x_i^{-1} \mid i \in V \setminus F]$. Since A is a localization of R , P_F is an associated prime of I^t if and only if $P_F A$ is an associated prime of $I^t A$. If $i \in F \setminus c(F)$, then i is adjacent to some vertex $j \notin F$. Hence $x_i = (x_i x_j) x_j^{-1} \in I A$. From this it follows that

$$I A = J A + (x_i \mid i \in F \setminus c(F)) A.$$

Let $B = k[x_i \mid i \in F]$. Let P the maximal homogeneous ideal of B and

$$Q = J B + (x_i \mid i \in F \setminus c(F)) B.$$

Then $P_F A = P A$ and $I^t A = Q^t A$. Since $A = B[x_i^{\pm 1} \mid i \in V \setminus F]$ is a Laurent polynomial ring over B , $P_F A$ is an associated prime of $I^t A$ if and only if P is an associated prime of Q^t . Since $B = S[x_i \mid i \in F \setminus c(F)]$, P is an associated prime of Q^t if and only if \mathfrak{n} is an associated prime of J^s for some $s \leq t$ [5, Lemma 2.1]. \square

By Theorem 3.3, to check the condition $\mathbf{n} \in \text{Ass}(J^s)$ we have to deal with dominating sets of $\Gamma_{c(F)}$. Such a set can be described in terms of Γ as follows.

Lemma 4.2. *Let F be a non-minimal cover of Γ . Then a subset $U \subseteq V$ is a dominating set of $\Gamma_{c(F)}$ if and only if F is minimal among the covers containing $N[U]$.*

Proof. By definition, U is a dominating set of $\Gamma_{c(F)}$ if and only if $U \subseteq c(F) \subseteq N[U]$. By the definition of the core, $U \subseteq c(F)$ if and only if every vertex of U is not adjacent to any vertex of $V \setminus F$. This means $F \supseteq N[U]$. It is also clear that $c(F) \subseteq N[U]$ if and only if every vertex $i \in F \setminus N[U]$ is adjacent to a vertex $j \in V \setminus F$. In other words, there exists an edge $\{i, j\}$ which is not covered by the set $F \setminus i$ or, equivalently, $F \setminus i$ is not a cover of Γ . Thus, U is a dominating set of $\Gamma_{c(F)}$ if and only if F is minimal among the covers containing $N[U]$. \square

Now we are able to deduce from Theorem 3.3 the following characterization of the associated primes of I^t in terms of Γ .

Theorem 4.3. *Let F be a cover of Γ . Then P_F is an associated prime of I^t if and only if F is a minimal cover or minimal among the covers containing $N[U]$ for some subset $U \subseteq V$ such that Γ_U is strongly non-bipartite with $\mu^*(\Gamma_U) < t$.*

Proof. We may assume that F is not a minimal cover. By Lemma 4.1, $P_F \in \text{Ass}(I^t)$ if and only if $\mathbf{n} \in \text{Ass}(J^s)$ for some $s \leq t$. By Theorem 3.3, this is equivalent to the condition that there is a dominating set U of $\Gamma_{c(F)}$ such that Γ_U is strongly non-bipartite with $\mu^*(\Gamma_U) < t$. Now we only need to apply Lemma 4.2 to obtain the assertion. \square

From Theorem 4.3 we immediately obtain the following results of Martinez-Bernal, Morey and Villarreal [24, Theorem 2.15].

Corollary 4.4. $\text{Ass}(I^t) \subseteq \text{Ass}(I^{t+1})$ for all $t \geq 1$.

By Theorem 4.3, the existence of an embedded prime of I^t depends only on the existence of a strongly non-bipartite induced graph Γ_U with $\mu^*(\Gamma_U) < t$. Hence can also easily deduce the following criterion for the existence of embedded primes of I^t , which was implicitly proved by Simis, Vasconcelos and Villarreal [30, Theorem 5.9] (see also [28, Lemma 3.5]).

Corollary 4.5. *I^t has no embedded primes if and only if Γ has no odd cycle of length $\leq 2t - 1$.*

Proof. If Γ has no odd cycles, then every induced graph Γ_U of Γ is bipartite. Hence I^t has no embedded primes for all $t \geq 1$. Therefore, we may assume that Γ has at least one odd cycle. If Γ has an odd cycle of length $\leq 2t - 1$ with the vertex set U , then Γ_U is strongly non-bipartite with $\mu^*(\Gamma_U) \leq t - 1$. If every odd cycle of Γ has length $\geq 2t + 1$, $|U| \geq c(2t + 1)$ for every strongly non-bipartite induced graph Γ_U , where c is the number of connected components of Γ_U . Hence $\mu^*(\Gamma_U) \geq (|U| - c)/2 \geq t$. \square

Another immediate consequence of Theorem 4.3 is that $P_F \in \text{Ass}^\infty(I)$ if and only if F is a minimal cover or F is minimal among the covers containing $N[U]$ for some subset $U \subseteq V$ such that Γ_U is strongly non-bipartite, which was proved in [5, Theorem 4.1] (implicitly for a connected graph) and [21, Corollary 3.6]. Below is a simple reformulation of this result.

For simplicity, let $\mathcal{C}(\Gamma)$ denote the set of the non-minimal covers F of Γ such that $\Gamma_{c(F)}$ is a strongly non-bipartite graph.

Corollary 4.6. *$P_F \in \text{Ass}^\infty(I)$ if and only if F is a minimal cover or $F \in \mathcal{C}(\Gamma)$.*

Proof. We may assume that F is not a minimal cover. By Lemma 4.2, $P_F \in \text{Ass}^\infty(I)$ if and only if there exists a dominating set U of $\Gamma_{c(F)}$ such that Γ_U is strongly non-bipartite. Since we may set $U = c(F)$, this is obviously equivalent to the condition that $\Gamma_{c(F)}$ is strongly non-bipartite. \square

In a more general setting, Sharp [29] asked the following question: Given a prime ideal $P_F \in \text{Ass}^\infty(I)$, can one identify an integer s such that $P_F \in \text{Ass}(I^t)$ for all $t \geq s$. If P_F is a minimal associated prime of I , we can set $s = 1$. If P_F is not a minimal associated prime of I , we can deduce from Theorem 4.3 that $s(\Gamma_{c(F)})$ is exactly the least number s with this property.

Theorem 4.7. *Let $F \in \mathcal{C}(\Gamma)$. Then $P_F \in \text{Ass}(I^t)$ if and only if $t \geq s(\Gamma_{c(F)}) + 1$.*

Proof. By definition, $s(\Gamma_{c(F)})$ is the minimum of $\mu^*(\Gamma_U)$, where U is a dominating set of $\Gamma_{c(F)}$ such that Γ_U is strongly non-bipartite. By Lemma 4.2, U is a dominating set of $\Gamma_{c(F)}$ if and only if F is minimal among the covers containing $N[U]$. Hence the assertion follows from Theorem 4.3. \square

By Corollary 4.5, I^t has no embedded primes for all $t \geq 1$ if and only if Γ is a bipartite graph. In this case, $\text{Ass}(I^t) = \text{Ass}(I^{t+1})$ for all $t \geq 1$. If Γ is a non-bipartite graph, then $\mathcal{C}(\Gamma) \neq \emptyset$ by Corollary 4.6. In this case, we immediately obtain from Theorem 4.7 the following formula for $\text{astab}(I)$, which is defined as the least number s such that $\text{Ass}(I^t) = \text{Ass}(I^{t+1})$ for all $t \geq s$.

Corollary 4.8. *Let Γ be a non-bipartite graph. Then*

$$(1) \quad \text{astab}(I) = \max\{s(\Gamma_{c(F)}) \mid F \in \mathcal{C}(\Gamma)\} + 1.$$

Remark 4.9. We do not know whether one can find a simpler formula for $\text{astab}(I)$ than (1). However, we can derive from (1) simpler upper bounds for $\text{astab}(I)$.

For $F \in \mathcal{C}(\Gamma)$ let $d(F)$ denote the set of vertices of degree ≥ 2 in $\Gamma_{c(F)}$. By Lemma 3.6, $s(\Gamma_{c(F)}) \leq \mu^*(\Gamma_{d(F)})$. Therefore,

$$(2) \quad \text{astab}(I) \leq \max\{\mu^*(\Gamma_{d(F)}) \mid F \in \mathcal{C}(\Gamma)\} + 1.$$

Let m be the number of the vertices of degree ≥ 2 and let $2t + 1$ be the smallest length of an odd cycle in Γ . For every initially odd ear decomposition \mathcal{E} of a connected strongly non-bipartite induced graph Γ_U , we have at most $|U| - (2t + 1)$ vertices of Γ_U which lies outside the first ear. Therefore, $\varphi^*(\mathcal{E}) \leq |U| - (2t + 1)$. From this it follows that $\varphi^*(\Gamma_{d(F)}) \leq m - c(2t + 1)$, where c is the number of connected

components of $\Gamma_{d(F)}$. Hence, $\mu^*(\Gamma_{d(F)}) = (\varphi^*(\Gamma_{d(F)}) + m - c)/2 \leq m + t - 1$ for all $F \in \mathcal{C}(\Gamma)$. Thus, (2) implies

$$(3) \quad \text{astab}(I) \leq m - t,$$

which is the upper bound found by Chen, Morey and Sung in [5, Corollary 4.3]. However, this bound is far from being optimal as observed in [21, Example 3.8].

Using Theorem 4.7 we can easily find the embedded primes of every I^t at the same time. In fact, we only need to look for the covers $F \in \mathcal{C}(\Gamma)$ and compute $s(\Gamma_{c(F)})$.

Example 4.10. Let Γ be the graph in Figure 5. There are only two covers F such that $\Gamma_{c(F)}$ is strongly non-bipartite: $F_1 = \{1, 2, 3, 4, 5\}$ with $c(F_1) = \{1, 2, 3\}$ and $F_2 = \{1, 2, 3, 4, 5, 6\}$ with $c(F_2) = \{1, 2, 3, 4, 5, 6\}$. We have $s(\Gamma_{c(F_1)}) = 1$ and $s(\Gamma_{c(F_2)}) = 2$. Thus, I^2 has only one embedded prime (x_1, \dots, x_5) and I^t , $t \geq 3$, have two embedded primes (x_1, \dots, x_5) and (x_1, \dots, x_6) . In this case, (3) is not sharp because $\text{astab}(I) = 3$, whereas $m - t = 6 - 1 = 5$.

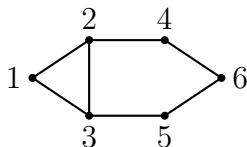


FIGURE 5.

5. MINIMAL BASES

For simplification we call a strongly non-bipartite graph Γ with $\mu^*(\Gamma) = s$ an s -base. We say that an s -base is *minimal* if it doesn't contain any other s -base with the same vertex set.

By Theorem 4.3, an ideal P_F is an associated prime of I^t if and only if F is a minimal cover or F is minimal among the covers containing the closed neighborhood of a minimal s -base for some $s \leq t - 1$. In particular, $\mathfrak{m} \in \text{Ass}(I^t)$ if and only if Γ has a dominating minimal s -base for some $s \leq t - 1$. Therefore, we only need to look for the minimal s -bases with $s \leq t - 1$ in order to find the associated primes of I^t .

It is easy to see that the triangle is the only 1-base. Hence we immediately obtain the following results of Herzog and Hibi [17] and Terai and Trung [33] on the associated primes of I^2 .

Theorem 5.1. [33, Theorem 3.8] *Let F be a cover of Γ . Then P_F is an associated prime of I^2 if and only if F is a minimal cover or F is minimal among the covers containing the closed neighborhood of a triangle.*

Theorem 5.2. [17, Theorem 2.1] [33, Theorem 2.8] *$\text{depth } R/I^2 > 0$ if and only if Γ has no dominating triangle.*

It is clear that an s -base is minimal if and only if each connected component is a minimal r -base for a suitable number r . For $s \geq 2$, a connected minimal s -base can be recursively constructed as follows.

Proposition 5.3. *Let Γ be a connected minimal s -base. Then either Γ is a cycle of length $2s + 1$ or Γ is the union of a connected minimal r -base Δ , $r < s$, with an ear of length $2(s - r)$ or $2(s - r) + 1$ meeting Δ only at the endpoints.*

Proof. Let \mathcal{E} be an initially odd ear decomposition of Γ with $\varphi(\mathcal{E}) = \varphi^*(\Gamma)$. If \mathcal{E} has only one ear, then this ear is an odd cycle of length n . By definition, $\mu^*(\Gamma) = (n - 1)/2 = s$. Hence $n = 2s + 1$. Since a $(2s + 1)$ -cycle is an s -base, Γ must be this cycle.

If \mathcal{E} has more than one ear, let C be the last ear of \mathcal{E} . Let Δ be the induced graph on the vertices of the other ears. Then Δ is non-bipartite because it contains the first odd ear of \mathcal{E} . Let \mathcal{F} be the initially odd ear decomposition of Δ obtained from \mathcal{E} by removing C . Let \mathcal{F}' be an arbitrary initially odd ear decomposition of Δ . If $\varphi(\mathcal{F}') < \varphi(\mathcal{F})$, we would have $\varphi(\mathcal{E}') < \varphi(\mathcal{E})$, where \mathcal{E}' denotes the ear decomposition of Γ obtained by adding C to \mathcal{F}' . This is a contradiction to the assumption $\varphi(\mathcal{E}) = \varphi^*(\Gamma)$. Thus, we must have $\varphi(\mathcal{F}') \geq \varphi(\mathcal{F})$. Hence $\varphi(\mathcal{F}) = \varphi^*(\Delta)$.

Let $r = \mu^*(\Delta)$. If Δ is not a minimal r -base, Δ contains another r -base Δ' with the same vertex set. Let Γ' denote the union of Δ' with the ear C . Choose \mathcal{F}' to be an initially odd ear decomposition \mathcal{F}' of Δ' with $\varphi(\mathcal{F}') = \varphi^*(\Delta')$. Since $\varphi(\mathcal{F}') = \varphi^*(\Delta') = \varphi^*(\Delta) = \varphi(\mathcal{F})$, we have $\varphi(\mathcal{E}') = \varphi(\mathcal{E})$. This implies $\varphi^*(\Gamma') \leq \varphi(\mathcal{E}) = \varphi^*(\Gamma)$. On the other hand, since Γ' has the same vertex set as Γ , every initially odd ear decomposition of Γ' is also an initially odd ear decomposition of Γ . Therefore, $\varphi^*(\Gamma') \geq \varphi^*(\Gamma)$. So we obtain $\varphi^*(\Gamma') = \varphi^*(\Gamma)$. This implies $\mu^*(\Gamma') = \mu^*(\Gamma)$, which contradicts the minimality of Γ as an s -base.

Let m be the number of vertices of Δ . If $m = n$, C must be an edge connecting two vertices of Δ . Hence $\varphi(\mathcal{F}) = \varphi(E)$. This implies $\varphi^*(\Delta) = \varphi^*(\Gamma)$. Therefore, Δ is an s -base with the same vertex set V , contradicting the minimality of Γ as an s -base. Thus, we must have $m < n$. Since $\varphi^*(\Delta) = \varphi(\mathcal{F}) \leq \varphi(\mathcal{E}) = \varphi^*(\Gamma)$, this implies

$$r = \mu^*(\Delta) = (\varphi^*(\Delta) + m - 1)/2 < (\varphi^*(\Gamma) + n - 1)/2 = \mu^*(\Gamma) = s.$$

Let ℓ be the length of C . Then

$$\begin{aligned} \ell &= n - m + 1 = (2s - \varphi^*(\Gamma) + 1) - (2r - \varphi^*(\Delta) + 1) + 1 \\ &= 2(s - r) - \varphi(\mathcal{E}) + \varphi(\mathcal{F}) + 1. \end{aligned}$$

If C is an even ear, $\varphi(\mathcal{E}) = \varphi(\mathcal{F}) + 1$. Hence $\ell = 2(s - r)$. If C is an odd ear, $\varphi(\mathcal{E}) = \varphi(\mathcal{F})$. Hence $\ell = 2(s - r) + 1$. \square

In general, the union of a connected minimal r -base with an ear of length $2(s - r)$ or $2(r - s) + 1$, $r < s$, is not an s -base.

Example 5.4. Let Δ be the graph of two disjoint triangles connected by an edge (or a path of length 2) as in Figure 6. Then Δ is a minimal 3-base (or 4-base). Let Γ be the union of Δ with a path of length 2 (or 3) which meets the two triangles at

different vertices than the endpoints of the first path. Then Γ is a minimal 3-base (or 4-base).



FIGURE 6.

The following lemma is a converse to Proposition 5.5.

Lemma 5.5. *Let Γ be the union of a connected minimal r -base Δ with a simple ear C of length $2(s - r)$ meeting Δ only at the endpoints, $r < s$. If C is an even cycle, then Γ is a minimal s -base.*

Proof. Let \mathcal{E} be an arbitrary initially odd ear decomposition of Γ . Since C is an even cycle meeting Δ only at the endpoint, \mathcal{E} can be decomposed as a union an initially odd ear decomposition \mathcal{F} of Δ with an ear decomposition of C . An ear decomposition of C consists of only C or of 2-cycles. If we assume furthermore that $\varphi(\mathcal{E}) = \varphi^*(\Gamma)$, then the ear decompositions of C and \mathcal{F} must have the smallest possible number of even ears. Therefore, the ear decomposition of C consists of only C and $\varphi(\mathcal{F}) = \varphi^*(\Delta)$. From this it follows that

$$\varphi^*(\Gamma) = \varphi(\mathcal{E}) = \varphi(\mathcal{F}) + 1 = \varphi^*(\Delta) + 1.$$

Let m denote the number of the vertices of Δ . Then $n = m + 2(s - r) - 1$ and $r = (\varphi^*(\Delta) + m - 1)/2$. Hence

$$\mu^*(\Gamma) = (n + \varphi^*(\Gamma) - 1)/2 = (m + 2(s - r) - 1 + \varphi^*(\Delta))/2 = s.$$

Assume that Γ contains another s -base Γ' with the same vertex set. Choose an initially odd ear decomposition \mathcal{E}' of Γ' with $\varphi(\mathcal{E}') = \varphi^*(\Gamma)$. Then \mathcal{E}' is also an initially odd ear decomposition of Γ . Let Δ' denote the induced subgraph of Γ' on the vertex set of Δ . As we have seen above, \mathcal{E}' can be decomposed as the union of an initially odd ear decomposition of Δ' and C . Hence Γ' is the union of Δ' and C . Since Γ' is a proper subgraph of Γ , Δ' is a proper subgraph of Δ . Moreover, we also have $\varphi^*(\Gamma) = \varphi^*(\Delta') + 1$. This implies $\varphi^*(\Delta') = \varphi^*(\Delta)$, contradicting the minimality of Δ as an r -base. So Γ must be a minimal s -base. \square

We do not know whether the above lemma is also valid for the case C is an odd cycle of length $2(s - r) + 1$. The above proof shows that it holds under the additional assumption $\varphi^*(\Delta) = \varphi(\Delta)$.

Using Proposition 5.3 and Lemma 5.5 we can easily find all minimal 2-bases. First, a connected minimal 2-base is either a pentagon or the union of a triangle with an ear of length 2 or 3. Note that an ear of length 2 can be a 2-cycle, which has to be viewed to be as an edge. If the ear is an edge or a 3-cycle, the resulting graph is a minimal 2-base. If the ear is a path of length 2 or 3, the resulting graph is not a minimal 2-base because it contains the union of the triangle with an edge or a pentagon, which has the same vertex set. A disconnected minimal 2-base must

be the union of two disjoint triangles because the triangle is the only 1-base. Thus, a minimal 2-base must be one of the four graphs in Figure 7.

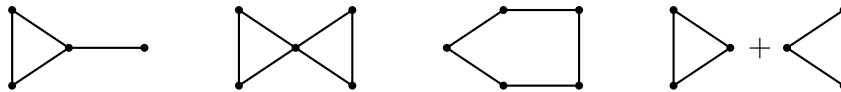


FIGURE 7.

From the above observation we immediately obtain the following results of Hien, Lam and Trung [21] on the associated primes of I^3 .

Theorem 5.6. [21, Theorem 4.4] *Let F be a cover of Γ . Then P_F is an associated prime of I^3 if and only if F is a minimal cover or F is minimal among the covers containing the closed neighborhood of a triangle or one of the four graphs of Figure 7.*

Corollary 5.7. [21, Corollary 4.5] *depth $R/I^3 > 0$ if and only if Γ has a dominating subgraph which is a triangle or one of the four graphs of Figure 7.*

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